Results

The linearized equations of motion given by Eq. (4) for a spinning symmetrical satellite in an elliptic orbit have been simulated on a digital computer. An example case where the orbital eccentricity $\epsilon = 0.5$, the inertia ratio $K = 1.0$, and the spin parameter $\alpha = -1.0$ is considered here. Using these values a Floquet analysis was done to determine the characteristic exponents and hence the stability of the uncontrolled motion. The exponents where

$$\rho_1 = 0.6718 \qquad \rho_3 = 0.0 + 0.3728i$$
$$\rho_2 = -0.6718 \qquad \rho_4 = 0.0 - 0.3728i$$

Since ρ_1 is positive the uncontrolled motion is unstable. The $F^{-1}(\tau)$ matrix was formed and $r_1(\tau)$ was determined and expressed in a Fourier series. The Fourier series for $r_1(\tau)$ has in this case a dc component given by

$$a_{0_1} = 0.6197$$

The control necessary therefore to change just ρ_1 is given by

$$u = [k_1, 0, 0, 0] F^{-1}(\tau) \bar{x} \tag{27}$$

where k_1 is given by $k_1 = (\rho_{1_{CL}} - \rho_1)/a_{0_1}$. A value of $k_1 = -2.0$ was chosen and a Floquet analysis of the closed-loop system yielded

$$\rho_{1_{CL}} = -0.5677 \qquad \rho_{3_{CL}} = 0.0 + 0.3728i$$
$$\rho_{2_{CL}} = -0.6718 \qquad \rho_{4_{CL}} = 0.0 + 0.3728i$$

These results agree to four places with the results predicted by the theory. Only $\rho_{1_{CL}}$ is different from the open-loop values and it is as predicted.

Conclusions

In summary, the modal control of a system of linear differential equations with periodic coefficients has been considered. General results for determining single-mode control have been derived. The general results have been demonstrated in controlling the attitude motion of a spinning symmetric satellite.

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Recursive Relationships for Body Axis Rotation Rates

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Introduction

In many algorithms and areas of analysis in flight dynamics and flight control systems, both body axis rotation rates and Euler angle rates are measured and/or used to control the motion of the vehicle. There are a number of Euler sequences (12 possible sets for three Euler rotations) any one of which can be used to describe a problem or may be required by hardware definition. Since there are quite a number of sequences possible and each equation relating the body axis rotation rates to the Euler rates is unique for that sequence, these relationships are not written out in the literature. The following analysis provides the engineer with a method of computing all these relationships and extends the equations to n Euler rotations. These equations lend themselves immediately to computer applications and can also be used to write out a particular Euler sequence when the computing equations are needed.

The transformation equation,

$$x = a\bar{x} \tag{1}$$

transforms vectors from the moving system ($\bar{x}$) to the inertial (stationary) system (x) at a particular instant in time. The transformation matrix can be constructed from n single-axis Euler rotations,

$$a = R_1 R_2 R_3 \ldots R_n \tag{2}$$

In most applications three or less Euler rotations are used, i.e., $n \leq 3$; however, the Space Shuttle IMU system uses a four-axis gimbal, thus requiring four Euler rotations to describe the transformation matrix from the navigation mounting base on the Space Shuttle to the stable member (inertial platform). The transformation matrix is orthogonal and has the properties,

$$a^T a = a a^T = I \tag{3}$$

where I is the unity matrix.

Differentiating Eq. (3),

$$\dot{a}^T a + a^T \dot{a} = \dot{a} a^T + a \dot{a}^T = 0 \tag{4}$$

then

$$\dot{a}^T a = -a^T \dot{a} \quad \text{and} \quad \dot{a} a^T = -a \dot{a}^T \tag{5}$$

or

$$a^T \dot{a} = -(a^T \dot{a})^T \tag{6}$$

which states that $a^T \dot{a}$ is skew symmetric. From other analyses we know that

$$\dot{a} = a\bar{W} \tag{7}$$

where $\bar{W}$ is the skew symmetric matrix of body axis rotation rates ($\bar{\omega}$), namely,

$$\bar{W} = \begin{bmatrix} 0 & -\bar{\omega}_3 & \bar{\omega}_2 \\ \bar{\omega}_3 & 0 & -\bar{\omega}_1 \\ -\bar{\omega}_2 & \bar{\omega}_1 & 0 \end{bmatrix} \tag{8}$$

Further if we differentiate Eq. (2),

$$\dot{a} = \dot{R}_1 R_2 R_3 \ldots R_n + R_1 \dot{R}_2 R_3 \ldots R_n + R_1 R_2 \dot{R}_3 \ldots R_n + \ldots + R_1 R_2 R_3 \ldots \dot{R}_n \tag{9}$$

Note that, like Eq. (7),

$$\dot{R}_n = R_n W_n \tag{10}$$

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where W_n is a skew symmetric matrix containing only the Euler rate about the nth axis for that rotation. For example, if the nth Euler rotation were about the y axis (pitch axis),

$$W_2 = \begin{bmatrix} 0 & 0 & \dot{\Theta} \\ 0 & 0 & 0 \\ -\dot{\Theta} & 0 & 0 \end{bmatrix} \tag{11}$$

where $\dot{\Theta}$ is the pitch rate. Substituting Eq. (10) into Eq. (9), the following recursive form results:

$$a^T \dot{a} = \bar{U}_n = \bar{W} \tag{12}$$

where

$$\bar{U}_n = R_n^T \bar{U}_{n-1} R_n + W_n \tag{13}$$

and

$$\bar{U}_0 = 0$$

The notation $\bar{U}_n$ here means a matrix formed as a result of n Euler rotations; for example, when $n=3$, three Euler rotations,

$$\bar{U}_3 = R_3^T (R_2^T W_1 R_2 + W_2) R_3 + W_3 \tag{14}$$

are derived from Eq. (13) in the following manner:

$$\bar{U}_1 = R_1^T \bar{U}_0 R_1 + W_1 = W_1$$

$$\bar{U}_2 = R_2^T \bar{U}_1 R_2 + W_2 = R_2^T W_1 R_2 + W_2$$

$$\bar{U}_3 = R_3^T \bar{U}_2 R_3 + W_3 = R_3^T (R_2^T W_1 R_2 + W_2) R_3 + W_3$$

Euler angles are measured from the inertial (stationary) coordinate frame to the instantaneous Euler axis position in the particular sequence that is being used. The rate of change of the Euler angles refers to rotations about one axis at a time, using the intermediate Euler axes as defined by the selected sequence. For instance, in the standard yaw-pitch-roll Euler sequence, the rate of change of the pitch angle is measured about a line where the Euler pitch is occurring and creates components of the rotation vector in the inertial frame as if only two rotations had occurred,

$$(\omega)_2 = R_1 R_2 \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} \tag{15}$$

where $R_1 = R_z(\Psi)$, a rotation about the yaw axis (z axis) and $R_2 = R_y(\Theta)$, a rotation about the pitch axis (y axis). The total inertial rotation vector is the sum of all rotation components resulting from the rates of change of each Euler angle, that is,

$$\omega = (\omega)_1 + (\omega)_2 + (\omega)_3 \tag{16}$$

for a three-axis Euler sequence. Writing out the sum resulting from n Euler angle rates of change,

$$\omega = R_1 S_1 + R_1 R_2 S_2 + R_1 R_2 R_3 S_3 + \ldots + R_1 R_2 R_3 \ldots R_n S_n \tag{17}$$

Where the S_n vectors are the instantaneous Euler rate vectors about the particular axis selected by the Euler angle sequence. By transforming the inertial rotation rate vector equation (17) into the body axis frame using Eq. (1), we have

$$\bar{\omega} = a^T \omega \tag{18}$$

Substituting ω from Eq. (17) and a^T from Eq. (2) we can write

$$\bar{\omega} = (R_n^T \ldots R_3^T R_2^T R_1^T)(R_1 S_1 + R_1 R_2 S_2 + R_1 R_2 R_3 S_3 + \ldots + R_1 R_2 R_3 .. R_n S_n) \tag{19}$$

which reduces to

$$\bar{\omega} = R_n^T \ldots R_3^T R_2^T S_1 + R_n^T \ldots R_4^T R_3^T S_2 + R_n^T \ldots R_5^T R_4^T S_3 + \ldots + S_n \tag{20}$$

This equation may be written in recursive form as follows:

$$\bar{\omega} = \bar{T}_n \tag{21}$$

$\bar{T}_n$ has the form,

$$\bar{T}_n = R_n^T \bar{T}_{n-1} + S_n \tag{22}$$

again $\bar{T}_0 = 0$.

The notation $\bar{T}_n$ means a vector formed as a result of n Euler rotations, where n is the number of Euler rotations, R_n is the nth single-axis rotation matrix, and S_n is a vector of the nth Euler derivative. For example, when $n=3$ (three Euler rotations), Eq. (22) becomes

$$\bar{\omega} = \bar{T}_3 \tag{23}$$

$$\begin{aligned} \bar{\omega} &= R_3^T \bar{T}_2 + S_3 \\ &= R_3^T (R_2^T \bar{T}_1 + S_2) + S_3 \\ &= R_3^T [R_2^T (R_1^T \bar{T}_0 + S_1) + S_2] + S_3 \\ &= R_3^T (R_2^T S_1 + S_2) + S_3 \end{aligned} \tag{24}$$

Hence, for any three Euler rotations,

$$\bar{\omega} = R_3^T (R_2^T S_1 + S_2) + S_3 \tag{25}$$

the body axis rotation rate vector as a function of the Euler rates and the single-axis Euler rotation matrices. As an example of the use of Eq. (21), using the standard yaw-pitch-roll aircraft system (this is a 3, 2, 1 Euler sequence), Eq. (25) becomes

$$\begin{bmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \\ \bar{\omega}_3 \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi & \cos\Phi \end{bmatrix}^T \times \left[\begin{bmatrix} \cos\Theta & 0 & \sin\Theta \\ 0 & 1 & 0 \\ -\sin\Theta & 0 & \cos\Theta \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ \dot{\Psi} \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} \right] + \begin{bmatrix} \dot{\Phi} \\ 0 \\ 0 \end{bmatrix} \tag{26}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix} \times \left[\begin{bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\Psi} \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} \right] + \begin{bmatrix} \dot{\Phi} \\ 0 \\ 0 \end{bmatrix} \tag{27}$$

which reduces to

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix} \begin{bmatrix} -\sin\Theta\ \dot\Psi \\ \dot\Theta \\ \cos\Theta\ \dot\Psi \end{bmatrix} + \begin{bmatrix} \dot\Phi \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

and finally,

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\sin\Theta\ \dot\Psi + \dot\Phi \\ \cos\Phi\ \dot\Theta + \sin\Phi\cos\Theta\ \dot\Psi \\ -\sin\Phi\ \dot\Theta + \cos\Phi\cos\Theta\ \dot\Psi \end{bmatrix} \tag{29}$$

the body axis rotation rates as a function of the Euler angles and their rates of change.[1]

Conclusion

The skew symmetric matrix of body axis rotation rates for any Euler sequence and any number of Euler rotations may be obtained from the recursive relationship,

$$\bar W = \bar U_n = R_n^T \bar U_{n-1} R_n + W_n \quad \text{with} \quad \bar U_0 = 0 \tag{30}$$

where R_n is the nth single-axis Euler rotation matrix and W_n is the skew symmetric matrix containing only the Euler rate about the nth axis for that rotation. This expression provides the engineer with a useful tool in analysis when the Euler angle rates are given. The body axis rotation rate vector can be expressed in the less complicated recursive relationship,

$$\bar\omega = \bar T_n = R_n^T \bar T_{n-1} + S_n \quad \text{with} \quad \bar T_0 = 0 \tag{31}$$

Again, R_n is the nth single-axis Euler rotation matrix and S_n is a vector containing only the nth Euler derivative. This expression for the body axis rotation rate vector can be used as a "cookbook" method to write out the computing equations when multiaxis, multisequence Euler rotations are required.

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